NOTES ON COCHLEAR FILTERING

J.S.C. van Dijk

1. INTRODUCTION

It is well-known that the basilar membrane has the ability to select frequencies from complex sound signals. A simple way to model this capacity is to represent the membrane by a row of tuned harmonic oscillators. In this work we shall assume that the oscillators are found at the real axis of a complex z -plane.
Because the membrane is surrounded by waterly fluids and rather weak structures, it is widely accepted that the driving force of a single oscillator is mainly of hydrodynamical origin. In consequence of this the equation of motion of a single resonator at the point x reads

$$m(x)\,\frac{d^2\eta(x,t)}{dt^2} + r(x)\,\frac{d\eta(x,t)}{dt} + k(x)\,\eta(x,t) = -\,2\,p(x,0,t)\ , \tag{1}$$

in which $\eta(x,t)$ is the deflection of the resonator at x. $m(x)$ and $k(x)$ respectively are the effective mass and stiffness at that point and $r(x)$ is a damping parameter. The right member of (1) denotes the pressure difference across the membrane, which for reasons of simplicity has been written as two times the pressure at the upperside of the membrane.
The relation between the velocity vector $\boldsymbol{v}$ of the surrounding medium and the pressure p within that medium is the (linear) equation of Euler

$$\rho\,\frac{\partial \boldsymbol{v}}{\partial t} = -\,grad\,p + \boldsymbol{F}\ ,$$

in which $\boldsymbol{F}$ stands for unknown additional forces per unit of volume. Near the membrane the viscosity of both the fluid and the weak surrounding structures tend to nullify a motion. We shall express these effects by an additional frictional force $\boldsymbol{F}=-\lambda\boldsymbol{v}$ in which λ is an effective frictional parameter.
We always assume that the membrane follows the motion of the medium. In that case the membrane velocity equals the velocity component of $\boldsymbol{v}$ normal to the membrane. Using the Euler equation, this is expressed by

$$\rho\,\frac{d^2\eta(x,t)}{dt^2} = -\,\frac{\partial p(x,0,t)}{\partial n} - \lambda\,\frac{d\eta(x,t)}{dt}\ , \tag{2}$$

where n is the normal to the membrane.
If the system has zero initial conditions, the Laplace transform of (1) -with respect to the time- reads

$$m\,(s^2 + 2\,\delta(x)\,s + \omega_0^2(x))\,\overline{\eta}(x,s) = -\,2\,\overline{p}(x,0,s)\ , \tag{3}$$

in which we use a bar to denote the transform of an original. In (3) the mass $m(x)$ has been considered as a constant $m(x) = m$ and in consequence of this $2\,\delta(x) = r(x)\,m$ and $\omega_0^2(x) = k(x)\,/\,m$.

The transform of (2) can be written as

$$(\rho\, s^2 + \lambda\, s\,)\,\bar{\eta}(x,s) = -\,\frac{\partial\,\bar{p}(x,0,s)}{\partial\, n}\ . \tag{4}$$

From (3) and (4) follows that

$$m\ \frac{s^2 + 2\ \delta(x)\ s + \omega_0^2(x)}{\rho\ s^2 + \lambda\ s}\ \frac{\partial\,\bar{p}(x,0,s)}{\partial\, n} = 2\,\bar{p}(x,0,s)\ . \tag{5}$$

Equation (5) has the shape of a homogeneous radiation condition. In section 3 we will show that (5) can be considered as a first order differential equation which is easy to solve.

However, in general the pressure near the upperside of the membrane will have a known part due to any forced fluid flow. In that case the unknown part of the pressure is the radiation pressure and (5) takes the shape of an inhomogeneous radiation condition.

In this article we shall restrict ourselves to the homogeneous case because the solution of an inhomogeneous equation can be generated from the solution of its homogeneous part. It will appear that the solution of the homogeneous equation (5) is appropriate to study transient behaviour of the basilar membrane deflection. For, if the pressure has been found from (5) the membrane deflection follows from (3).

2. DAMPING PARAMETERS

In (3) the effective damping of the resonator at x is determined by $\delta(x)$. According to Van Dijk (1986) it is useful to write for $\delta(x)$

$$\delta(x) = \sin\varepsilon_1\ \omega_0(x)\,, \tag{6}$$

in which ε_1 is a small positive number. Because $\delta(x)$ is proportional to $\omega_0(x)$ all resonators have the same quality factor.

The second damping parameter models an effective damping due to the internal friction of both the cochlear fluids and the weak structures of a stratum near the membrane.

It can be shown that in case of fluid viscosity the number λ depends on frequency (Van Dijk, 1976). In order to model some of the properties in a simplified manner, we propose to consider λ proportional to the circular frequency ω. Therefore, we put

$$\frac{\lambda}{\rho} = \varepsilon_2\ \omega\ , \tag{7}$$

in which ε_2 is a small positive constant.

Equation (5) possesses singular points. From (6) and (7) follows that these singularities are found at the left of the imaginary axis of the complex s - plane.
Mainly for reasons of convenience, we shall rotate this plane a quarter of a turn to the left. This is accomplished by

$$w = is \; . \tag{8}$$

Using (6), (7) and (8) a calculation shows that (5) can be written as

$$-\frac{m}{2\rho}\,\frac{\left(\frac{\omega_0(x)}{w} + e^{-i\varepsilon_1}\right)\left(\frac{\omega_0(x)}{w} - e^{+i\varepsilon_1}\right)}{1 + i\,\frac{\varepsilon_2\,\omega}{w}}\,\frac{\partial \bar{p}(x,w)}{\partial n} = \bar{p}(x,w)\;, \tag{9}$$

where $\bar{p}(x,w) = \bar{p}(x,0,w)$ .The left member of (9) shows that the singularities of equation (9) are

$$\begin{aligned} w_1 &= -\,\omega_0(x)\,exp\,(+\,i\,\varepsilon_1) \\ w_2 &= \omega_0(x)\,exp\,(-\,i\,\varepsilon_1) \\ w_3 &= -\,i\,\varepsilon_2\,\omega \qquad , \end{aligned} \tag{10}$$

and are found in the under halfplane Im $w \leqslant 0$. In the next section we will specify $\omega_0(x)$ at the real axis of the complex z -plane, so that it will be easy to relate the z -plane to the complex w -plane.

3. TOWARDS A MEMBRANE MODEL.

In auditory theory it is often assumed that the relation between the resonance frequency of subsequent oscillators and the length parameter x along the membrane is an exponential decreasing function of x . According to this idea $\omega_0(x)$ can be written as

$$\omega_0(x) = Q\,exp(-\,ax\,/\,2\,)$$

in which Q and a are positive constants. Usually a membrane model starts at $x = 0$, so that Q is the resonance frequency of the first resonator. Gummer and Johnstone (1983) showed that there is no evidence to support this notion. They showed that the resonance frequency is approximately a decreasing linear function of the place at the membrane. Therefore, we replace the exponential relation by the linear function

$$\omega_0(x) = Q\,/\,x\,/ \quad , -\infty < x < 0 \; . \tag{11}$$

Here we assume that the basilar membrane coincides with the negative real axis of the z -plane. Its high frequency side points to minus infinity and its low frequency part is found near the origin.

Equation (9) can be written as

$$\frac{\partial \ln \bar{p}(x,w)}{\partial n} = v \left[\frac{1}{\frac{\omega_0(x)}{w} + e^{-i\varepsilon_1}} - \frac{1}{\frac{\omega_0(x)}{w} - e^{+i\varepsilon_1}} \right] .$$

The constant v is given by

$$v = \frac{\rho}{m \cos \varepsilon_1} \left(1 + i \frac{\varepsilon_2 \, \omega}{w} \right) . \qquad (12)$$

Insertion of (11) for $-\infty < x < 0$ yields

$$\frac{\partial \ln \bar{p}(x,w)}{\partial n} = v \left[\frac{1}{\frac{Q\,x}{w} + e^{+i\varepsilon_1}} - \frac{1}{\frac{Q\,x}{w} - e^{-i\varepsilon_1}} \right] . \qquad (13)$$

We shall look at (13) as an equation which has been defined along the whole real axis of the z -plane. However, in the applications we have to confine ourselves to the negative part of the real axis.
For reasons of convenience we first assume that we are dealing with real values of w.
A more general extension will be given in section 4.
The right member of (13) shows the following singular points.

$$z_1 = -\frac{w}{Q}\, e^{+i\varepsilon_1} \qquad\qquad z_2 = \frac{w}{Q}\, e^{-i\varepsilon_1} \qquad (14)$$

We note that the place of z_1 and z_2 in the z -plane depends on the sign of w. Therefore, we distinguish among the following cases.

case 1 : $w = \; |w|$ (15)

if w is positive and real.

case 2 : $w = -|w|$ (16)

if w is negative and real.

Case 1.

From (12), (13) and (15) follows that

$$\frac{\partial \ln \bar{p}(x,w)}{\partial n} = v' \left[\frac{1}{\frac{Q\,x}{|w|} + e^{+i\varepsilon_1}} - \frac{1}{\frac{Q\,x}{|w|} - e^{-i\varepsilon_1}} \right] , \qquad (17)$$

in which

$$v' = \frac{\rho}{m \cos \varepsilon_1} \left(1 + i \frac{\varepsilon_2 \omega}{|w|}\right) . \tag{18}$$

The right member of (17) shows that the abscissae of the singular points comprise a weighting factor $Q / |w|$. We shall assume that the unknown ordinates include the same weighting factor. In that case it is natural to replace in (17) the normal n by $Q y / |w|$. Therefore, we put

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial \left(\frac{Q y}{|w|} \right)} . \tag{19}$$

From (14) and (15) we see that the place of the singularities in (17) is below the real axis. In consequence of this, the pressure in (17) can be conceived as the boundary value of a function $\bar{p}(z,w)$ which is analytic in the upperhalf plane Im $z \geqslant 0$. Even if ε_1 tends to zero this notion remains valid. In that case the singular points are automatically taken in the boundary. Therefore, we propose to consider

$$\bar{p}(x,w) = \bar{p}(x + i\,0, w) . \tag{20}$$

From (17), (19) and (20) follows that at the real axis $\bar{p}(z,w)$ follows from

$$i \ln \bar{p}(z,w) = v' \ln \frac{Q z + |w| e^{+i\varepsilon_1}}{Q z - |w| e^{-i\varepsilon_1}} + C_1 . \tag{21}$$

The function $\bar{p}(z,w)$ is regular in the upper halfplane and bounded for $z \to \infty$. At the negative real axis $\bar{p}(z,w)$ is a solution of the membrane equation (17). The constant C_1 can be used to fit $\bar{p}(z,w)$ to a prescribed pressure at a given point of the membrane. Due to its analytic behaviour in the upper halfplane, $\bar{p}(z,w)$ satifies Laplace's equation. That means $\bar{p}(z,w)$ is the pressure in an incompressible inviscous flow.

Case 2.

Insertion of (16) in (12) and (13) yields

$$\frac{\partial \ln \bar{p}(x,w)}{\partial n} = \overline{v'} \left[\frac{1}{\frac{Q x}{|w|} + e^{-i\varepsilon_1}} - \frac{1}{\frac{Q x}{|w|} - e^{+i\varepsilon_1}} \right] , \tag{22}$$

where $\overline{v'}$ is complex conjungate to (18). According to (14) and (16) in this case the singularities of (22) are points in the upper halfplane. Therefore, we shall look at $\bar{p}(x,w)$ as

$$\bar{p}(x,w) = \bar{p}(x - i\,0, w) . \tag{23}$$

Using again (19) we find from (22) and (23) that

$$- i \ ln \ \bar{p}(\bar{z},w) = \overline{v'} \ ln \ \frac{Q \ \bar{z} + / \ w \ / \ e^{-i\varepsilon_1}}{Q \ \bar{z} \ - \ / \ w \ / \ e^{+i\varepsilon_1}} + C_2 \ . \tag{24}$$

If $C_2 = \overline{C_1}$, it can be shown that at the real axis $\bar{p}(z,w) = \overline{\bar{p}(\bar{z},w)}$.

4.PROPERTIES OF AN ARBITRARY POINT OF THE MEMBRANE

For (21) the point $z = - / x /, \infty < x < 0$, is a point of the membrane. However, $\bar{p}(z,w)$ is restricted to (15). Insertion of (15) and $z = - / x /$ in (18) and (21) yields

$$ln \ \bar{p}(x,w) = - i \ v \ \ ln \ \frac{Q \ / \ x \ / \ - w \ e^{+i\varepsilon_1}}{Q \ / \ x \ / \ + \ w \ e^{-i\varepsilon_1}} \ , \quad - \infty < x < 0 \ . \tag{25}$$

v is defined according to (12) .We arrive at the same expression after substitution of the $z = - / x /, - \infty < x < 0$, and (16) in (24). This means that (25) needs no additional requirements to the sign of w. It is not difficult to recognize (25) as a special case of

$$ln \ \bar{p}(x,w) = - i \ v \ \ ln \ \frac{\omega_0(x) \ - \ w \ e^{+i\varepsilon_1}}{\omega_0(x) \ + \ w \ e^{-i\varepsilon_1}} \ , \tag{26}$$

in which $\omega_0(x)$ is given by (11) and w is a complex variable.
From (8), (26) and (12) readily follows that the pressure $\bar{p}(x,w)$ can be written as a function of s .The result is

$$ln \ \bar{p}(x,s) = \ i \ v \ \ ln \ \frac{s \ - \ i \ \omega_0(x) \ e^{+i\varepsilon_1}}{s \ + \ i \ \omega_0(x) \ e^{-i\varepsilon_1}} \ , \tag{27}$$

in which an unimportant constant has been neglected.The number v reads

$$v = \frac{\rho}{m \ cos \ \varepsilon_1} \ (\ 1 + \frac{\varepsilon_2 \ \omega}{s} \ ) \ .$$

We are not able to invert (27). Therefore, we investigated $\bar{p}(x,s)$ at the frequency axis of the complex s - plane. At this axis $s = i \ \omega$ so that v is reduced to

$$v = \frac{\rho}{m \ cos \ \varepsilon_1} \ (\ 1 \ - \ i \ sgn \ (\omega) \ \varepsilon_2 \ ) \ . \tag{28}$$

The shape which is basic to (27) reads

$$ln \ \bar{p}(s) = \ i \ v \ \ ln \ \frac{s + tan \ \varepsilon_1 - i}{s \ + \ tan \ \varepsilon_1 \ + \ i} \ , \tag{29}$$

and is found after application of the well-known time scaling rule with scaling factor

$$\omega_{\varepsilon_1}(x) = \omega_0(x) \cos \varepsilon_1 \ . \tag{30}$$

It should be noted that (28) does not depend on scaling. $\bar{p}(s)$ -as has been defined by (29)- is the scaled transfer function of an arbitrary point of the membrane.
Properties of $\bar{p}(s)$ in the frequency domain are easily found after introduction of polar coordinates according to

$$\begin{aligned} s + \tan \varepsilon_1 - i &= r_1 \exp (i \, \phi_1) \\ s + \tan \varepsilon_1 + i &= r_2 \exp (i \, \phi_2) \ . \end{aligned} \tag{31}$$

From (28), (29) and (31) we find that at the axis $s = i\,\omega$

$$\ln / \bar{p}(\omega) / = \mu \ (\phi_2(\omega) - \phi_1(\omega)) + \mu \ sgn(\omega) \ \varepsilon_2 \ \ln \frac{r_1(\omega)}{r_2(\omega)} \tag{32}$$

and

$$\arg(\bar{p}(\omega)) = \mu \ \ln \frac{r_1(\omega)}{r_2(\omega)} - \mu \ sgn(\omega) \ \varepsilon_2 \ (\phi_2(\omega) - \phi_1(\omega)) \tag{33}$$

where

$$\phi_{1,2}(\omega) = \arctan \frac{\omega \mp 1}{\tan \varepsilon_1}$$

and (34)

$$r_{1,2}(\omega) = \sqrt{\tan^2 \varepsilon_1 + (\omega \mp 1)^2} \ .$$

The constant μ is defined by

$$\mu = \frac{\rho}{m \cos \varepsilon_1} \ . \tag{35}$$

It can be shown that the frequency characteristics of the deflection $\bar{\eta}(\omega)$ which belong to (32) and (33) have the shape

$$\ln / \bar{\eta}(\omega) / = \ln / \bar{p}(\omega) / - \ln \ r_1(\omega) \ r_2(\omega) \tag{36}$$

and

$$\arg(\bar{\eta}(\omega)) = \arg(\bar{p}(\omega)) - (\phi_1(\omega) + \phi_2(\omega)) \ . \tag{37}$$

In (36) the first term of the right member corresponds to (32). The first term of the right member of (37) is given by (33).

The next figures show amplitude and phase characteristics of pressure .

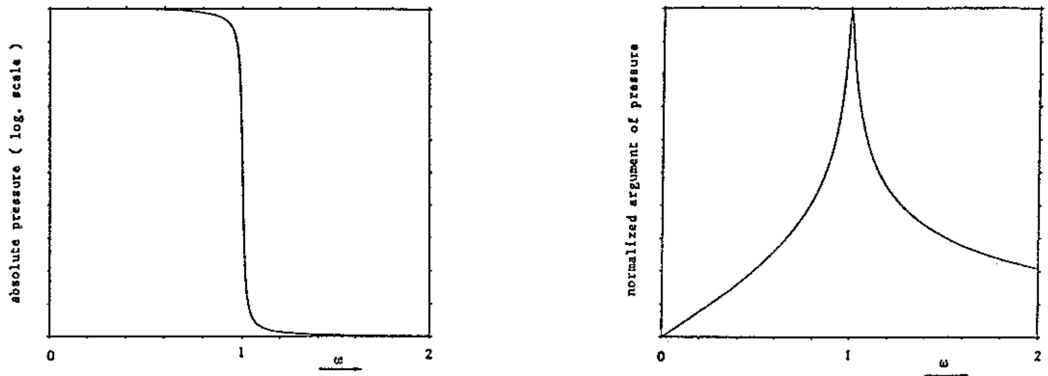

Fig. 1. Amplitude and phase characteristics of the pressure according to (32) and (33). At resonance the amplitude of the pressure is an almost discontinuous function and the phase function has a weak singularity. Parameter values for the damping constants are: $\varepsilon_1 = 0.0001$ and $\varepsilon_2 = 0.01$.

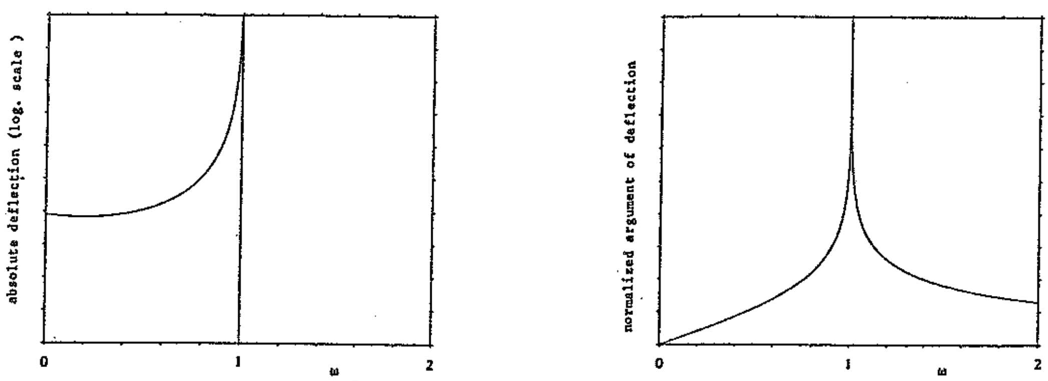

Fig. 2. Amplitude and phase characteristics of the deflection according to (36) and (37). Parameter values for the damping constants are: $\varepsilon_1 = 0.0001$ and $\varepsilon_2 = 0.01$. Sharpness near resonance is controlled by ε_1 . The parameter ε_2 mainly controls the slope before resonance. At resonance the phase function shows the same weak singular behaviour as the phase function of the pressure.

ε_1 controls the sharpness at resonance. This parameter models internal friction of a single oscillator. ε_2 can be considered as an overall damping parameter due to surrounding structure and fluid friction .
The main objection against these characteristics is the singular behaviour of the phase at resonance. This problem will be solved in a next article.

The rate of change of phase at $\omega = 0$ determines the delay time . From (33) and (35) follows that

$$\lim_{\omega \to 0} \frac{d\, arg(\bar{\eta}(\omega))}{d\omega} \approx \lim_{\omega \to 0} \frac{d\, arg(\bar{p}(\omega))}{d\omega} ,$$

where we again assumed that ε_1 and ε_2 are sufficient small. This means that the delay time of the deflection is mainly controlled by the delay time of the pressure. A simple calculation shows

$$\lim_{\omega \to 0} \frac{d\, arg(\bar{p}(\omega))}{d\omega} \approx -2\,\mu \tag{38}$$

which corresponds to the derivative of $arg(\bar{p}(\omega))$ in the lossless case ($\varepsilon_1 = 0$, $\varepsilon_2 = 0$). The delay time τ is defined as minus the derivative of the phase at $\omega = 0$ .Thus we find that

$$\tau = 2\,\mu$$

In the present shape τ denotes a scaled time with scaling factor (30). After re-scaling of the time we arrive at

$$\tau(x) = \frac{2\,\mu}{\omega_0(x)\, cos\, \varepsilon_1} \,. \tag{39}$$

This time can be written as

$$\tau(x) = 2\, \frac{\rho}{\sqrt{m\, k(x)}} \,, \tag{40}$$

where we inserted (35) in (39) and used the well-known relation$\omega_0^2(x) = k(x)\,/\,m$. Because $\varepsilon_1 << 1$ we have cos $\varepsilon_1 \approx$ 1.

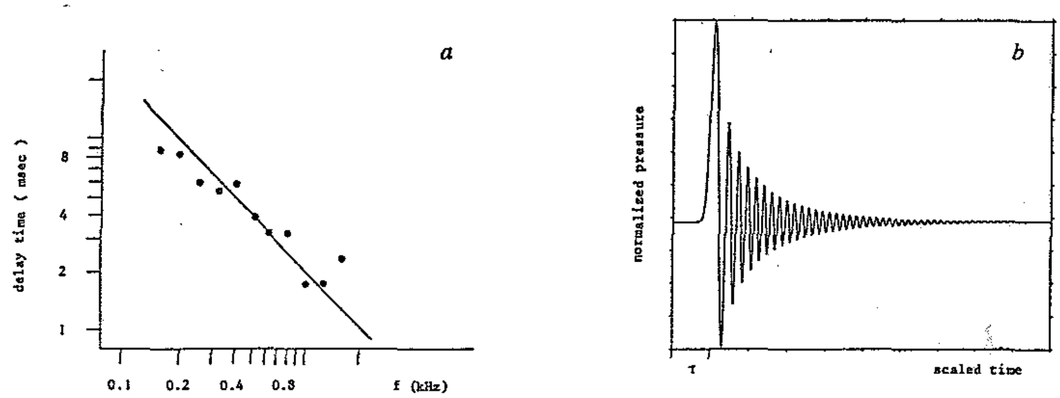

Fig. 3. The straight line in 3a represents the delay time of the pressure according to (39). The dots are cochlear nerve delay times for the chinchilla as a function of the 'best frequency' of a nerve-fibre (Ruggero, 1987). The scaled delay time is clearly visible in the impulse response of the pressure which follows from (32) and (33).

Fig. 3a is a plot of $\tau(x)$ as a function of $f_0(x) = \omega_0(x)\,/2\pi$. The straight line represents (39) with $\mu = 20$. This value follows from (35) with usual parameter values $\rho = 1$ and m = 0.05.The dots in fig. 3 are cochlear nerve delay times for the chinchilla as a function of the 'best frequency' of a nerve-fibre (Ruggero, 1987).

In Fig. 3b the impulse pressure response at an arbitrary point of the membrane has been shown. This function can be found from (29) after insertion of $s = i\,\omega$ and application of a numerical inverse Fourier transform. Fig. 3a strongly suggests that the main contribution to the cochlear nerve delay time equals the travel time of an implusive travelling pressure wave along the membrane. Fig. 3b. shows that the scaled delay time τ is an adequate measure for the calm before the storm.

5. CONCLUSIONS

In the cochlea the interaction between the basilar membrane, the surrounding weak structures and the cochlar fluids leads to an inhomogeneous radiation condition for the pressure at the membrane. We studied the homogeneous case because its solution generates the solution of the inhomogeneous case. This restriction leads to a description of some basic transient properties of the membrane behaviour.
Two different damping parameters have been introduced. The first one models the internal resistance of a mechanical membrane oscillator. In general this kind of damping is rather small. A second damping parameter is used to model the effective resistance of adjacent weak structures and the internal friction of the cochlear fluids. We assume that this kind of damping is greater than the internal resistance of a single oscillator. In consequence of this, sharpness at resonance can be modelled very well even at moderate values of the second kind of damping.
Response curves for pressure and deflection have been determined. It appears that near resonance phase characteristics show singular behaviour. The delay time -which follows from phase characteristics near the origin- is mainly determined by the delay time of the pressure impulse along the membrane. This time almost equals the cochlear nerve delay time.

In cochlear hydrodynamics, conditions of the kind (5) play an essential role at describing cochlear boundary value problems. As a radiation condition (5) always leads to rather complicated mathematical models. In order to find a solution of (5) at a fixed frequency, usually the whole boundary value problem or an equavalent integral equation is solved or approximated by means of (semi)numerical techniques. For an overview we refer to parts from Allen et al (1986) or De Boer (1984).
We showed that in the two-dimensional case it is easy to find an exact solution for the boundary condition. Application of well-known scaling rules, straightforwardly leads to transfer properties of the pressure near the membrane and the deflection of an arbitrary point of the membrane. The main scaling factor is approximately the resonance frequency of the membrane oscillator at an arbitrary point. In the present shape the transfer function is suited to develop digital recursive membrane models using known techniques from signal analysis

6. REFERENCES

Allen, J.B., Hall, J.L., Hubbard, A., Neely, S.T. and Tubis, A. Eds (1986). Peripheral Auditory Mechanisms. Lecture Notes on Biomathematics 64. Springer-Verlag. Berlin, Heidelberg, New York, Tokio.

Boer, E. de (1984). Auditory physics. Physical principles in hearing theory II. Physics Reports. Vol. 105, no.3

Dijk, J.S.C. van (1976). "On the hydrodynamics of the inner ear. Theoretical part. A mathematical model", Acustica 35, 190-201.

Dijk, J.S.C. van (1986). "The complete solution of the basilar membrane condition". In Peripheral Auditory Mechanism. Eds. J.B. Allen, J.L. Hall, A. Hubbard, S.T. Neely and A. Tubis, Lecture Notes in Biomathematics 64, Springer-Verlag Berlin, Heidelberg, New York, Tokio .

Gummer, A.W. and Johnstone, B.M. (1983). "State of stress within the basilar membrane: a re-evalutation of the basilar membrane misnomer", Hearing Research 12, 353-366.

Ruggero, M.A. and Rich. N.C. (1987). "Timing of spike initiation in cochlear afferents: dependence on site of innervation", J Neurophysiol. 58, 379-403.