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BEYOND RESONANCE[1]

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Abstract

In the frequency domain the boundary condition for the pressure at the basilar membrane has the shape of a homogeneous equation. We solved this equation as an equation in the complex plane and conceived the solution as a 'free field' description of the pressure. This analytical description models a sink at the point of resonance. After that we applied the method of images and constructed the pressure distribution in a strip-like model of the cochlea. It appeared that after the point of resonance the phase tends to constant values. However, for most applications it is sufficient to restrict ourselves to the analytical solution.

1 Introduction

Until the observations of Rhode (1971; 1973), the common opinion on the motion of the basilar membrane as a result of pure tone stimulation was the concept of a travelling wave along the membrane. According to that notion the motion of the membrane resembles a progressive wave which travels from the stapes to the helicotrema. During its travel the wave reaches a maximum in a small region of the membrane around the point of resonance. After that region the amplitudes of the wave rapidly diminishes. However, after resonance it is questionable whether the wave still travels or not. Von Békésy (1928)introduced the concept of a travelling wave after his early model observations. Later model studies and observations in preparations of the cochlea (Von Békésy, 1960) seemed to support this notion. Rhode showed that the concept of a travelling wave only holds true as far as the point of resonance. After that point amplitudes of the membrane motion are almost negligibly small. Besides, after resonance the successive points of the membrane perform a motion in almost the same phase.

In consequence of Von Békésy's observations it is not surprising that even in early attempts to describe the motion of the basilar membrane, the intention was present to model the pressure in the surrounding fluid, even beyond resonance, as a travelling

[1]This is a slightly adapted part from Chapter 4 in the Ph. D. thesis of Van Dijk (2001).

wave. Ranke (1931, 1942) was one of the firsts who introduced a travelling wave concept in a two-dimensional model study. He noticed that the pressure in the cochlear fluid has to obey Laplace's equation and pointed to the fact that the general solution of this equation can be written as the sum of two arbitrary functions, each of which depends on one of the complex conjugate co-ordinates z and $\bar{z}$. Essentially, the shape of Ranke's solution is an expression of the kind $\exp(c(x+iy))$ at the membrane axis $y=0$ of the complex plane. The constant c is a complex quantity. His idea was to fit this constant so that for successive points of the membrane the pressure and the velocity at the membrane obey the definition of the local impedance. From his work follows that just after resonance the amplitude of the pressure strongly diminishes and that near resonance the local wavelengths are relatively short.

Siebert (1974) re-investigated Ranke's question. He proposed a solution for the pressure in a two-dimensional box-like model of the cochlea. Under a short-wave assumption he arrived at solutions for the motion of the basilar membrane. The relevant parts of the solutions show waves that travel towards the point of resonance. Apart from technical details of his analysis, the numerical implementation of these waves points to singular behaviour near resonance.

In a review article on cochlear models Schroeder (1975) expresses his dissatisfaction on the short wave approximations with the sentence "This kind of modelling is out". Fortunately, de Boer (1979, 1984) studied the short wave case again. In addition to an improvement of Siebert's mathematical description, he pointed to the physical phenomenon that the point of resonance at the membrane locally acts as a sink for the energy which is present both at the membrane and in its fluid-like environment. This is an indication that the influence of resonance at the membrane not only determines what happens at the membrane but dictates what happens in the surrounding fluid too. And vice versa. This corresponds to classical ideas from complex function analysis that behaviour is determined by the presence of singularities. There is no reason at all to go away from those ideas and, what is more, some phase characteristics in recent observations (Ruggero *et al.*, 1997) seem to underline that indeed near resonance the wave at the membrane travels towards the point of resonance.

The behaviour of the wave motion near the point of resonance is a typical example of a local phenomenon. This behaviour takes place both at and near the membrane. Then the question arises: is it really necessary to introduce boundary value problems for the whole cochlea in order to find this behaviour? The rhetorical character of this sentence implies the answer. In this contribution we will elucidate this.

We start with the almost trivial observation that it is sufficient to determine the hydrodynamic pressure at the membrane. Once the pressure is known, the motion of the membrane readily follows.

Let us consider a general shape for an oscillating pressure wave at the membrane. This wave can be written as $p(x,t)=\hat{p}(x,\omega)\cos(\omega t+\varphi(x,\omega))$. The (real) amplitude of the wave is $\hat{p}(x,\omega)$ and $\varphi(x,\omega)$ is the phase function. ω is the frequency of the oscillations and x the length parameter along the membrane. Any undulatory behaviour in the x - direction follows from $\varphi(x,\omega)$. When the slope of this function is negative the wave travels to the right. When the slope is positive the wave travels in the opposite direction.

As has been noted by Ranke, the general solution of Laplace's equation can be written as the sum of two functions. Each of these functions depends on one of the complex conjugate co-ordinates z and $\bar{z}$. Therefore, we shall conceive the travelling wave that obeys Laplace's equation at the membrane as a limiting function of two complex conjugate functions $p_1(\bar{z},t)$ and $p_2(z,t)$ so that

$$2p(x,t) = \lim_{y \to 0}\left(p_1(\bar{z},t) + p_2(z,t)\right) , \tag{1}$$

in which

$$\lim_{y \to 0} p_1(\bar{z},t) = \hat{p}(x,\omega)\exp(+i(\omega t + \varphi(x,\omega)))$$

and

$$\lim_{y \to 0} p_2(z,t) = \hat{p}(x,\omega)\exp(-i(\omega t + \varphi(x,\omega))) .$$

In (1) the function $p_1(\bar{z},t)$ depends on positive[2] frequencies and $p_2(z,t)$ on negative ones. In this chapter we shall assume that the stiffness along the membrane is a decreasing function along the membrane. Then, as follows from Van Dijk (2001; chapter 3 section 3.4.3), we may expect that in the lossy case $p_2(z,t)$ is determined by a singularity just below the membrane axis and $p_1(\bar{z},t)$ by its reflection with respect to this axis. Therefore, in order to find the pressure completely, we will distinguish between the upper plane approximation $z = x + i0$ for $p_2(z,t)$ and the lower plane approximation $\bar{z} = x - i0$ for $p_1(\bar{z},t)$. The distance of both singularities to the real axis is determined by the magnitude of the damping. In consequence of this, in the lossless case both singularities coincide at the membrane axis. However, because the lossless case is the limiting case of the lossy one, the notion to distinguish between an upper- and a lower-plane approximation remains conserved.

In the present work we will mainly restrict ourselves to the lossless case but distinguish between both approximations.

First we will shortly derive the key of the problem, namely the basilar membrane condition as an equation for the pressure. The solution of this equation in the place domain will be given and discussed. The pressure according to the direct solution is an analytical function.

This function is not odd with respect to the normal direction to the membrane but shows a lot of good properties. At the end of section 2 we propose a solution for the pressure which is odd with respect to this normal direction at the membrane. It will appear that the analytical solution of this section forms part of this odd solution. This

[2] Positive frequencies are points of the positive frequency axis of the complex s plane; negative frequencies are points at the negative part of this axis.

aspect of the problem renews the discussion on the question whether the analytical solution represents the kernel of the general solution of the problem.

2 A discontinuity as far as resonance

In this section we shall pay attention to properties of the basilar membrane that directly follow from the point of resonance as a mathematical singularity. Our starting point is the linear equation of motion

$$\frac{\partial^2 u_{mn}}{\partial t^2} + \omega_o^2 u_{mn} = -\frac{2}{m} p \ . \tag{2}$$

Here, u_{mn} is the deflection of the basilar membrane perpendicular to the membrane and $-2p$ is the pressure difference across the membane. m is the mass of the membrane per unit of area. As usual ω_0^2 is the stiffness of the membrane normalised to the mass m. The deflection and the pressure depend on the place x at the membrane and the time t. At the membrane u_{mn} coincides with the normal component u_n of the fluid deflection and the pressure there equals the fluid pressure.

Both quantities have to obey the same Euler equation normal to the membrane. Therefore, in the absence of additional forces, we have in the linear case the additional requirement

$$\frac{\partial^2 u_{mn}}{\partial t^2} = -\frac{1}{\rho}\frac{\partial p}{\partial n} \ . \tag{3}$$

The density of the fluid is ρ and n the normal to the membrane. The difference between (2) and (3) reads

$$\frac{1}{\rho}\frac{\partial p}{\partial n} - \frac{2}{m} p = \omega_0^2 u_{mn} \ . \tag{4}$$

Because the membrane deflection equals the fluid deflection normal to the membrane it is superfluous to use special indices. Therefore, we will omit the indices $_m$ and $_{mn}$.

The present equations are linear. Then it is attractive to apply the technique of Laplace transforms. For the sake of convenience, we shall assume that we only deal with existing transforms (Spiegel, 1965). Moreover, we assume that all initial conditions are zero. In that case the transform of equation (2) is

$$\left(s^2 + \omega_0^2\right)\bar{u} = -\frac{2}{m}\bar{p} \ , \tag{5}$$

in which $\bar{u}$ and $\bar{p}$ are the transformed deflection and pressure, respectively.

The transforms of (3) and (4) are

$$s^2\bar{u} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial n}$$

and

$$\frac{1}{\rho}\frac{\partial \bar{p}}{\partial n} - \frac{2}{m}\bar{p} = \omega_0^2\bar{u} \ ,$$

respectively. The deflection $\bar{u}$ can be eliminated from the last two equations. This leads to the equation for the pressure in the shape

$$\left(1+\frac{\omega_0^2}{s^2}\right)\frac{\partial \bar{p}}{\partial n} = \frac{2\rho}{m}\bar{p} \ .$$

The present equation determines the topic of this section. It appears to be useful to rewrite this equation in the equivalent shape

$$\frac{\partial \ln \bar{p}}{\partial n} = \mu s\left(\frac{1}{s+i\omega_0}+\frac{1}{s-i\omega_0}\right) \ , \tag{6}$$

in which the constant μ is defined by

$$\mu = \frac{\rho}{m} \ . \tag{7}$$

We make the substitution

$$w = is \ , \tag{8}$$

that maps the complex s plane on the complex w plane by a rotation of the s plane over a quarter of a turn to the left. The frequency axis of the s plane is mapped on the real axis in the w plane. Positive values at this axis correspond to points at the negative frequency axis in the s plane. The negative part of the real w axis is the image of the positive frequency axis in the s plane. As a result of this substitution (6) takes a shape which is rather convenient to work with. The equation has the shape

$$\frac{\partial \ln \bar{p}}{\partial n} = \mu w\left(\frac{1}{\omega_0+w}-\frac{1}{\omega_0-w}\right) \ . \tag{9}$$

Now we are in a position to specify the place of the basilar membrane in the z plane and to define ω_0 along the membrane. We assume that the basilar membrane coincides with the negative real axis of a complex z plane. Its high frequency part starts at minus infinity. The membrane ends at the origin. In this work we will only consider the simple case that the resonance frequency varies linearly along the membrane. Thus here $\omega_0(x) = -Q\,x$. The positive constant Q can be met with time scaling. Therefore it is sufficient to put $Q = 1$. Then we have

$$\omega_0(x) = -x \ , \ -\infty < x \le 0 \ . \tag{10}$$

We first restrict ourselves to positive real values of w. Insertion of (10) in (9) yields

$$\frac{\partial \ln \overline{p}}{\partial n} = \mu \left(\frac{1}{\frac{x}{w}+1} - \frac{1}{\frac{x}{w}-1} \right) \ ; \ -\infty < x \le 0 \ , \ y = 0 \ . \tag{11}$$

The right member of (11) possesses singular points on the real axis of the z plane. The place of these points follows from

$$\frac{x}{w} = \mp 1 \ .$$

For our problem the singularity with the minus sign is of special importance because it represents the point of resonance at the membrane. At the left side of this point the stiffness dominates. Between resonance and the origin the stiffness is of secondary importance and there we could put a zero condition for the pressure. However, this problem will be postponed at this stage and solved at the end of this section. Here, we will first look for some basic properties that follow from (11).

In (11) the abscissa which determines resonance comprises the scaling factor $1/w$. This factor determines the actual place of resonance at the membrane as a function of the frequency. Because we conceive (11) as an equation in the complex $z = x + iy$ plane, the unknown ordinate has been scaled with the same factor. In that case it is 'natural' to replace the normal n by y/w. Therefore we put

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial \left(\frac{y}{w} \right)} \ .$$

In consequence of this, the pressure $\overline{p}$ in (11) can be considered as a function at the real axis of a scaled z plane. We shall denote this plane as the $\zeta = \xi + i\eta$ plane, so that

$$\zeta = \frac{z}{w} \ . \tag{12}$$

Next we conceive $\overline{p}$ as the limiting case of a function of ζ, so that on the real axis it holds that $\overline{p}(\xi) = \overline{p}(\xi + i0)$.

Because at this axis

$$\frac{\partial \bar{p}}{\partial \eta} = i \frac{d\bar{p}}{d\zeta} \ ; \ -\infty < \xi < \infty \ , \ \eta = 0 \ ,$$

equation (11) can be integrated immediately. The result reads

$$i \ln \bar{p}(\zeta) = \mu \ln \frac{\zeta + 1}{\zeta - 1} \ ; \ -\infty < \xi < \infty \ , \ \eta = 0 \ , \qquad (13)$$

where we omitted for the sake of convenience the constant of integration.

It is well known from classical applications (for instance Spiegel, 1964) that the right member of (13) represents an orthogonal co-ordinate system in the complex ζ plane. This system determines the amplitude and phase of the pressure at the real axis of the ζ plane. In order to find these quantities it is customary to express the right member of (13) in terms of local polar co-ordinates with respect to the points $\zeta = \mp 1$. Here we confine ourselves to write in (13) the logarithm as

$$\ln\left(\frac{\zeta + 1}{\zeta - 1}\right) = \ln\left|\frac{\zeta + 1}{\zeta - 1}\right| + i \arg\left(\frac{\zeta + 1}{\zeta - 1}\right) .$$

Then, 'constant-amplitude curves' for the pressure belong to the family

$$\arg\left(\frac{\zeta + 1}{\zeta - 1}\right) = \alpha \ , \qquad (14)$$

in which α is a constant that varies from curve to curve. The members of this family are the circles

$$\xi^2 + (\eta + \cot\alpha)^2 = \frac{1}{\sin^2 \alpha} .$$

'Constant-phase curves' constitute the family

$$\ln\left(\frac{\zeta + 1}{\zeta - 1}\right) = \beta \ , \qquad (15)$$

in which β is a constant. This constant varies from member to member of the family. The phase family consists of the circles

$$(\xi - \coth\beta)^2 + \eta^2 = \frac{1}{\sinh^2 \beta} .$$

Both families together constitute the well-known meshwork of Apollonius, which has been given in Figure 1.

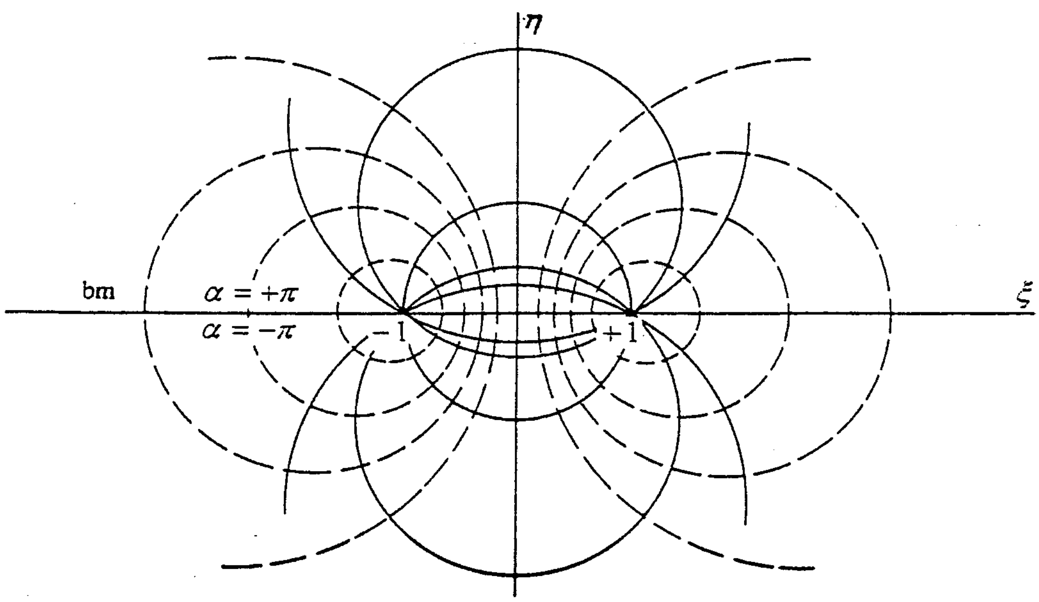

Figure 1. The classical co-ordinate system that is build up from circles of Apollonius and their orthogonal trajectories. This system determines the amplitude and phase of the pressure according to (13). The same system can be used as an intrinsic co-ordinate system in a boundary value problem that leads to an odd behaviour of the pressure with respect to the membrane axis. This behaviour is one of the properties to which the pressure at a cross section of the membrane has to be subjected.

The solid lines are the iso-amplitude curves. Dashed curves are curves of equal phase.

In the upper half-plane the range of α is $0 \le \alpha \le \pi$. At the unit circle we have $\alpha = \pi/2$. For $\alpha = \pi$ and $\alpha = 0$ the corresponding circles have been degenerated and coincide with the real axis.

In the lower half-plane the range of α is $-\pi \le \alpha \le 0$. In this case too, both equal signs correspond to circles, which have been degenerated and coincide with the real axis. The value $\alpha = -\pi/2$ corresponds to the unit circle.

The range of the phase β varies according $-\infty < \beta < +\infty$. In the left half-plane β is negative, whereas in the right half-plane its sign is positive. The imaginary axis represents the degenerated circle $\beta = 0$. The points -1 and $+1$ are the degenerated circles $\beta = -\infty$ and $\beta = +\infty$, respectively.

According to (13), (14) and (15) the pressure can be written as

$$\bar{p}(\alpha,\beta) = \exp(\mu(\alpha - i\beta)) \ , \tag{16}$$

where α and β follow from (14) and (15) and Fig. 1. The negative real axis represents the basilar membrane. That part of the membrane axis, which extends from minus infinity up to and including the point of resonance, i.e. the point $\xi = -1$, is a cut in the complex ζ plane. Across this cut the pressure is discontinuous.

At the upper-side of this axis and near the point of resonance the amplitude of the pressure shows discontinuous behaviour. At the left side of resonance the amplitude equals $\exp(\mu\pi)$ and diminishes suddenly to 1 at the other side of resonance. This implies that the difference between the amplitude levels before and after resonance equals $20\mu\pi \log e\ dB$. The constant μ is defined in (7). When $c\ g\ s$ units are used a typical value of this quantity is 20. Then the difference between the levels exceeds 500 dB. Ranke (1942) was the first who noticed this discontinuous behaviour. After him, De Boer (1979, 1984) has given a considerably better description of this phenomenon.

The quantity $-\beta$ is the phase of the pressure at the membrane. β tends to minus infinity as the distance to the point of resonance tends to zero. The slope of $-\beta$ is positive at the left side of resonance and negative at the other side. In consequence of this, the pressure at both sides of resonance represents a wave that travels always to the point of resonance. Moreover, de Boer (1979) showed from this near resonance behaviour that the point of resonance actually models a 'sink' for the corresponding energy in the direct environment of the point of resonance. According to Lighthill (1978, 1981) this behaviour is a typical example of what in hydrodynamics is known as critical layer absorption.

From the large difference between the levels before and after resonance it follows that the pressure is effectively zero between the point of resonance and the origin of the ζ plane. When the pressure in that region should be zero indeed we expect, at least on physical grounds, that in that region the pressure is odd with respect to the membrane axis. The present solution for the pressure fails to describe this property even approximately. Direct inspection shows that the pressure is inverted when values of α and β at the lower side of the membrane axis are used.[3]

In order to solve this 'imperfection', we define a boundary value problem in which α and β are considered as functions which determine a curvilinear co-ordinate system in the ζ plane. At the left side of resonance we prescribe at the membrane the present shape of the solution for the pressure. Between resonance and the origin of the plane we put the new boundary condition $p = 0$ at the membrane axis.

Next, we reflect these boundary conditions with respect to the imaginary axis of the ζ plane. Then we arrive at a boundary value problem which in terms of α and β reads

$$\begin{aligned} &\Delta\overline{p} = 0 && 0 < \alpha < \pi \ , \ -\infty < \beta < +\infty \ ; \\ &\overline{p} = e^{\mu\pi}\left(\cos\mu\beta - i\sin\mu\beta\right) && \alpha = \pi \ , \ -\infty < \beta < +\infty \ ; \\ &\overline{p} = 0 && \alpha = 0 \ , \ -\infty < \beta < +\infty \ . \end{aligned} \tag{17}$$

The solution for this problem is

$$\overline{p} = \frac{e^{\mu\pi}}{\sinh\mu\pi}\sinh\mu\alpha\left(\cos\mu\beta - i\sin\mu\beta\right) \ . \tag{18}$$

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[3] The inversion is most easily demonstrated when in the expression z^{-i}, $-i$ is rashly replaced by i.

It is easy to verify that this solution complies with the terms of problem (17).

Let us extend the present solution over the whole α,β plane. Then, because α is odd with respect to the line of symmetry $\alpha = 0$, it is readily seen that (18) has a second important property

$$\bar{p}(\alpha,\beta) = -\bar{p}(-\alpha,\beta) \tag{19}$$

that holds true even at the left-hand side of the point of resonance. Therefore, as a result of this kind of modelling, the pressure difference across the membrane in the region where the stiffness dominates is twice the pressure at the upper-side of the membrane too. This odd behaviour, which includes the membrane as a discontinuity as far as resonance, gives the possibility to construct an infinite strip-like model so that the normal derivative of the pressure vanishes at boundaries at a distance h from the basilar membrane.

We will carry out this process with a simplified expression for (18). In order to find the simplification we first return to the original expression for the pressure (13). Let us restrict ourselves to the part of the pressure that is determined by the singularity at $\zeta = -1$ and let us translate this part to the origin of the complex plane. This is accomplished when $\zeta + 1$ is replaced by ζ. The approximation reads

$$i \ln \bar{p} \approx \mu \ln \zeta \; .$$

It is our aim to express the pressure in polar co-ordinates (r,φ). However, in the right member of this expression the point $\zeta = 0$ is a branch point. Because this point is the image of the original singular point $\zeta = -1$, it follows from Fig. 1 that the right choice for φ is $-\pi < \varphi \le \pi$. Besides, this choice ensures that the principal branch of the logarithm corresponds to the usual one.

Therefore, we write the pressure as

$$\bar{p} \approx \exp(\mu\varphi - i\mu \ln r) \; ,$$

with $-\pi < \varphi \le \pi$. Note that the same expression can be found from (16) when α is replaced by φ and β by $\ln r$. With the same substitutions, we readily find from (18) that the desired simplification for the pressure reads

$$\bar{p} = \frac{e^{\mu\pi}}{\sinh \mu\pi} \sinh \mu\varphi \left(\cos \mu \ln r - i \sin \mu \ln r\right) \; . \tag{20}$$

The next step is the construction of a strip-like model of a cochlear scala. We will do this in the presence of damping. In Van Dijk (2001, section 3.3) we studied the influence of the damping on the place of the mathematical point(s) of resonance. When damping is present, there is a small distance between the physical point of resonance on the basilar membrane and the mathematical point of resonance ($\zeta = 0$) near the membrane. This distance equals the damping constant ε.

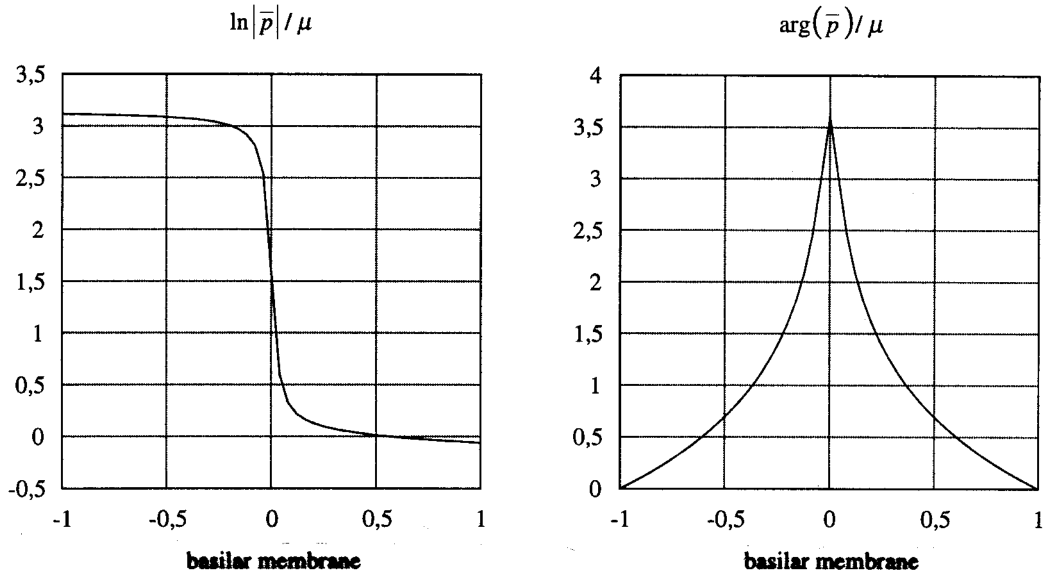

Figure 2. Amplitude and phase characteristics of the pressure according to (20) at a fixed distance $\eta = 0.025$ from the real axis of the $\zeta = \xi + i\eta$ plane. The distance $\eta = 0.025$ models damping. The amplitude of the pressure is almost discontinuous at the point of the resonance, the point $\xi = 0$. When η should be equal to zero, the amplitude possesses a non-removable discontinuity at the origin. The behaviour that is caused by the discontinuity is moderated when the distance to the membrane increases. The function $\ln|r|$ determines the phase of the pressure. The phase is (weak) singular at the origin. At the point of resonance r is equal to 0.025. This small value means that near the point of resonance the phase is almost singular. When the distance to the origin increases this behaviour becomes much weaker. The dimensions of $\ln|\bar{p}|/\mu$ are Nepers per unit of μ . The constant is given by (7). The dimension of μ is 1/cm. The unit of $\arg(\bar{p})/\mu$ is radian times centimetre.

This offers the opportunity to simulate damping by considering the pressure at a distance $\eta = \pm\varepsilon$ from the real axis of the plane. Figure 2 shows properties of the pressure according to (20) at a distance $\eta = 0.025$ from the real axis.

The left plot in this figure shows the level $\ln|\bar{p}|$ per unit of μ . The units of this quantity are the same as the units of (14). The right plot is the argument of (20) per unit of μ . Units of this part are comparable with the units of (15).

In order to construct a strip-like model of a cochlear scala, we applied the method of images to the ‘free field’ pressure according to (20). Let h be the scaled scala height of the cochlear scalae and let ε be a damping coefficient. The scaling factor follows from (12). Then we are interested in the pressure distribution in the infinite strips $-\infty < \xi < +\infty$; $\varepsilon < \eta < h + \varepsilon$ and $-\infty < \xi < +\infty$, $-\varepsilon < \eta < -h - \varepsilon$, so that the straight lines $\eta = \pm(h + \varepsilon)$ parallel to the real axis represent hard walls. In order to reach this, we shifted the expression (20) over a distance $2(h + \varepsilon)$ upwards and reflected the result with respect to the ordinate axis. Next we translated (20) downwards over a distance $\eta = 2(h + \varepsilon)$ and again reflected the resulting function. In this process we neglected all values of the pressure between the lines $\eta = \pm\varepsilon$.

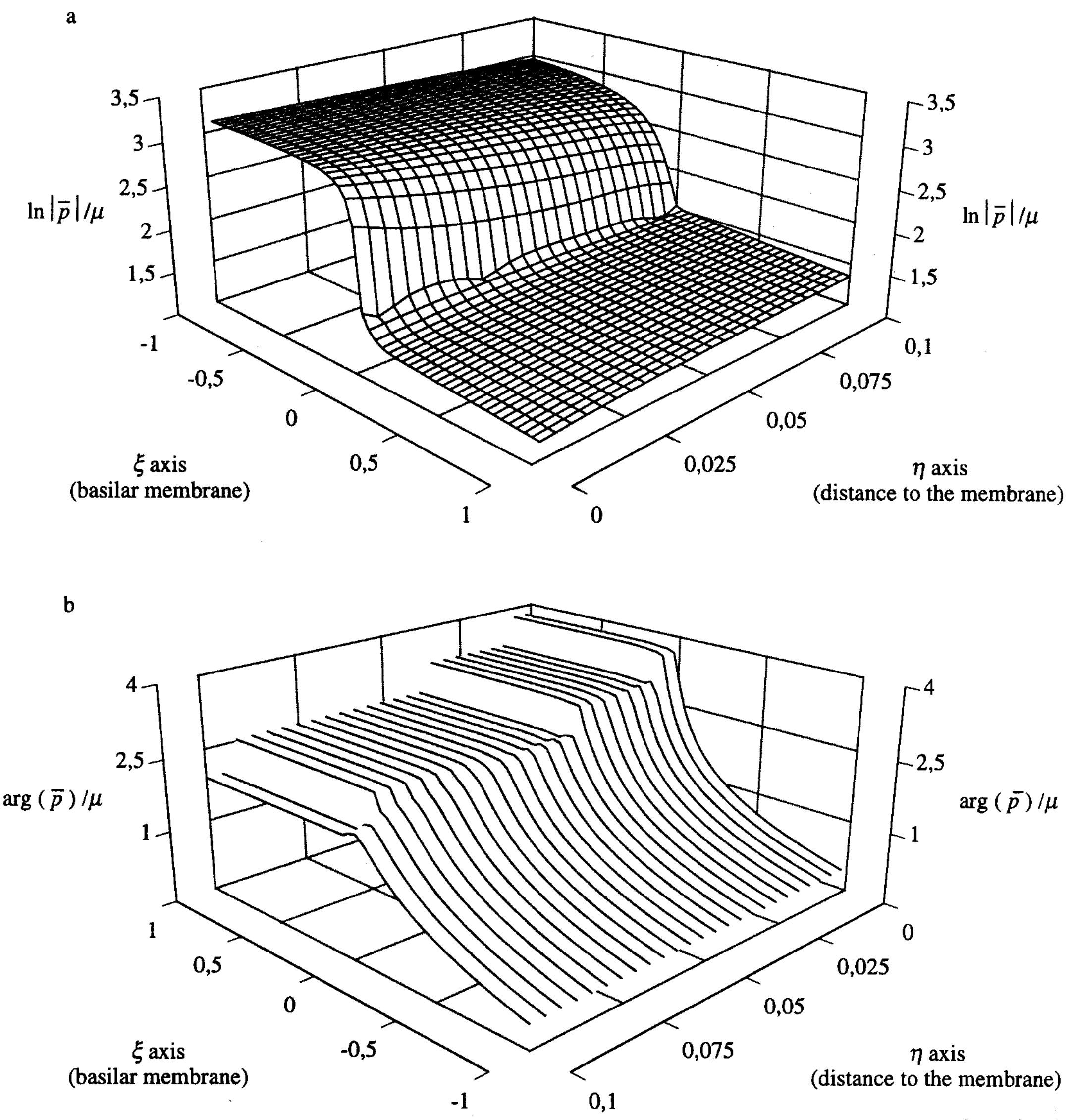

Figure 3. Amplitude and phase distribution in a strip-like model of a cochlear scala. The height (scaled) h of the strip is 0.1. The damping coefficient ε equals 0.025. The numerical value of μ is 10. In order to construct this strip-like model we applied the method of images and shifted and reflected the pressure (20) fifty times, equally distributed over the positive and the negative η direction. The dimension of h is determined by (12). a. The relative pressure distribution in the length direction ξ of the strip as a function of the distance η to the basilar membrane. For a fixed value of ξ the pressure is maximal at the basilar membrane. This corresponds to the notion that when the distance to the membrane increases it is as if the damping coefficient increases. Units of the relative pressure $\ln|\bar{p}|/\mu$ are Nepers per unit of μ. The dimension of μ follows from (7) and is 1/cm. b. Plot of the argument of the pressure, per unit of μ, in the strip-like scala. Each curve shows the phase at a fixed distance η to the basilar membrane. For negative values of ξ the phase is approximately constant. This leads to plateau-like phase behaviour throughout the strip. In this region the 'jumps' between some neighbouring curves correspond to 2π radians. Therefore, the jumps do not disturb plateau-like behaviour. The dimensions of $\arg(\bar{p})/\mu$ are radians times centimetres. Note that in the second plot both the direction of ξ and η are reversed compared with the direction of the abscissa and ordinate in the first plot.

The process of shifting and mirroring can be extended at infinity. The superposition of the terms as a result of this process yields a periodic pressure distribution in the η direction. The period is $4(h+\varepsilon)$.

Figure 3 shows the distribution of the amplitude and the phase of the pressure in a part of the strip $-\infty < \xi < +\infty$; $\varepsilon < \eta < h+\varepsilon$. We shifted and mirrored the pressure (20) fifty times; twenty-five times upwards and twenty-five times downwards.

For the sake of convenience we ultimately replaced $\eta - \varepsilon$ by η, so that the basilar membrane is found at $-\infty < \xi < +\infty$; $\eta = 0$ and the hard wall at $\eta = h$.

The upper figure shows the amplitude of the pressure distribution in a part of the strip at both sides of the point of resonance. Resonance takes place at the point $(0,0)$. The abscissa ξ is parallel to the length direction of the strip. The ordinate η determines the distance to the membrane. The basilar membrane is the line $\eta = 0$. Each curve at a fixed value of the ordinate shows an amplitude distribution at a fixed distance to the membrane.

The lower part of the figure shows the phase of the pressure, for fixed values of the ordinate η, as a function of ξ. In this part we reversed the direction of both the abscissa and the ordinate. The figure makes clear why this was done.

The figure shows that extreme values of both the amplitude and the phase are found at the boundaries of the strip.

When ξ is negative, the slope of the phase consists of two parts. Parallel to the abscissa the slope is positive at every point of this characteristic. In addition to this, it appears that there is a small negative slope parallel to the ordinate. Near the hard wall this last slope is almost negligible. This slope slightly 'increases' in the direction of the membrane. Clearly, both parts of the characteristics represent a travelling wave. The first one travels towards the line $\xi = 0$ and the second one towards the membrane $\eta = 0$.

For positive values of ξ it is as if the phase parallel to the abscissa tends towards a constant. However, a careful inspection shows that there remains always a small negative slope towards the axis $\xi = 0$. However, the steepness of the slope depends on the number of reflections that we applied as well as on h. Parallel to the ordinate the slopes are slightly decreasing whereas of course near the hard wall $\eta = 0.1$ this slope tends to vanish. Therefore in this region too, there is a pressure wave towards the line $\xi = 0$ and to the membrane $\eta = 0$.

Figure 4 shows plots of the amplitude and the phase of the pressure at the membrane for different values of h. Again the number of shifts and reflections is fifty and equally distributed over the positive and the negative η direction. The plots of the pressure level show that the level difference between the levels before and after the point of reference is about half the difference between the same levels in Fig. 2. However, even this level difference is very large. Because of this property, it holds that the pressure after the point of resonance is effectively equal to zero. In all figures the range of the phase is broadly the same.

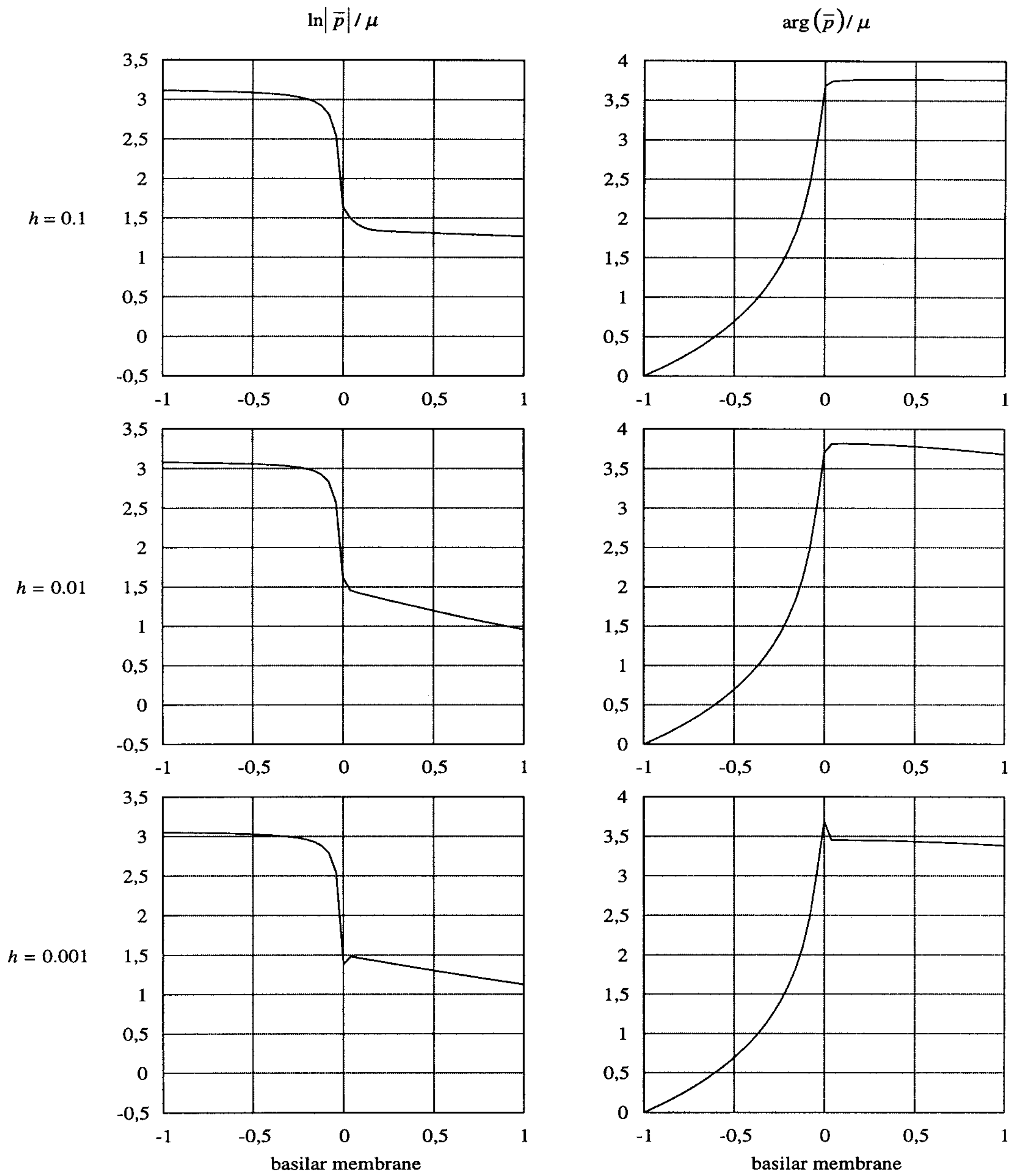

Figure 3. Amplitude and phase distribution in a strip-like model of a cochlear scala. The height (scaled) h of the strip is 0.1. The damping coefficient ε equals 0.025. The numerical value of μ is 10. In order to construct this strip-like model we applied the method of images and shifted and reflected the pressure (20) fifty times, equally distributed over the positive and the negative η direction. The dimension of h is determined by (12). a. The relative pressure distribution in the length direction ζ of the strip as a function of the distance η to the basilar membrane. For a fixed value of ζ the pressure is maximal at the basilar membrane. This corresponds to the notion that when the distance to the membrane increases it is as if the damping coefficient increases. Units of the relative pressure $\ln|p|/\mu$ are Nepers per unit of μ. The dimension of μ follows from (7) and is 1/cm. b. Plot of the argument of the pressure, per unit of μ, in the strip-like scala. Each curve shows the phase at a fixed distance η to the basilar membrane. For negative values of ζ the phase is approximately constant. This leads to plateau-like phase behaviour throughout the strip. In this region the 'jumps' between some neighbouring curves correspond to 2π radians. Therefore, the jumps do not disturb plateau-like behaviour. The dimensions of $\arg(p)/\mu$ are radians times centimetres. Note that in the second plot both the direction of ξ and η are reversed compared with the direction of the abscissa and ordinate in the first plot.

In addition to this, it holds that before the point of resonance the shapes of the phase in Fig. 2 and Fig. 4 are similar to each other. Therefore, in most applications it is sufficient to restrict ourselves to properties of the pressure that follow from the behaviour of the pressure before the point of resonance.

Essentially, the present solution for the pressure follows from (11). We studied this equation for positive real values of w. When w is negative, i.e. $w = -|w|$, we expect that the pressure $\bar{p}$ that follows from this equation is the complex conjugate of (13). This requirement has only been satisfied when the pressure $\bar{p}$ at the negative real axis is conceived as a function of the complex conjugate co-ordinate $\bar{\zeta}$. In consequence, we have to conceive the pressure $\bar{p}$ at the membrane for negative values of w as $\bar{p}(\xi) = \bar{p}(\xi - i0)$. Then, when the same way of reasoning of the first part of this section is followed, the ultimate result reads

$$-i \ln \bar{p}(\bar{\zeta}) = \mu \ln \frac{\bar{\zeta} + 1}{\bar{\zeta} - 1} \text{ , with } \bar{\zeta} = \frac{Q}{|w|} \bar{z} \text{ .} \tag{21}$$

It can be shown that the pressure that follows from (21) is indeed complex conjugate to the pressure according to (13). We shall not work out a problem similar to (17) for the conjugate pressure. This is not necessary. When in (18) i is replaced by $-i$, the complex conjugate counterpart is found immediately and possesses the required symmetry properties.

3 Discussion

In mathematics it is common knowledge that the behaviour of a solution for a problem in which singularities are present follows from the characteristics of the problem near those points. This is an indication to restrict ourselves to only those singular points that are close to the basilar membrane. In consequence of this we paid attention to a simplified problem for the pressure near the basilar membrane so that the properties that follow from the point of resonance can be easily found. This approximation corresponds to the notion that only these points are the physically relevant points of the problem.

The place of a point of resonance near the membrane depends on both frequency and damping. The latter parameter is responsible for the distance of the mathematical point of resonance to the basilar membrane. However, the sign of the frequency under consideration determines whether the point of resonance is found at the upper side of the membrane or at the lower side (Van Dijk, 2001; section 3.4.3). Therefore, in order to study near-resonance effects adequately in the place domain, we have to distinguish between positive and negative frequencies and in consequence of this to an upper-plane or a lower-plane approximation to the problem. In the lossless case the points of resonance coincide at the membrane axis. However, because the lossless case is the limiting case of the lossy one, the distinction between an upper- and a lower-plane approximation must remain conserved. The upper-plane and the lower-plane approach to the membrane axis yield expressions for the complex amplitudes for harmonic vibrations proportional to $\exp(+i\omega t)$ and $\exp(-i\omega t)$. The respective amplitudes are complex conjugate to each other. This is an indication that the proposed distinction is the only correct way to solve our problem.

For commonly used values of the damping the distance of the mathematical point of resonance to the basilar membrane is rather small. Then it follows that the effective distance from the stapes to the point of resonance on the membrane is a scaled length parameter. The scaling factor is the frequency under consideration. This follows from the singularities of the problem. When the normal to the membrane is chosen in agreement with this behaviour, it follows that the membrane condition can be straightforwardly integrated.

In the last few decades, only Siebert (1974) and De Boer (1979; 1984) paid attention to properties of the basilar membrane condition that are related to the present approach. Siebert (1974) started to study this problem, partially in a numerical way. De Boer (1979) considered the lossless and the lossy case. After the application of some analytical means, he argued that in the direct vicinity of the point of resonance the solution for the pressure represents a wave that travels towards the point of resonance.

There is no clear evidence whether the pressure after resonance must represent a wave that travels towards the point of resonance or not. In general, the amplitude of the membrane motion after the point of resonance is very small. In consequence of this, it is almost impossible to make hard decisions on this question. The current opinion is that beyond the point of resonance, all points of the basilar membrane perform a motion in almost the same phase. In that case it is as if the analytical solutions of this research note bear the intrinsic imperfection of a travelling wave after resonance. However, very recent measurements (de Boer and Nuttall, personal communication) seem to confirm that at both sides of the point of resonance there exist travelling waves. In that case the point of resonance is actually a sink.

The analytical solution of the membrane condition shows that after the point of resonance the level of the pressure is small. This behaviour can be approximated effectively when we assume that the pressure between the point of resonance and the helicotrema is zero. Then it is impossible to express the pressure both at the membrane and in the plane in terms of analytical functions. This is a typical property of an analytical function that follows from the principle of analytical continuation (see for instance Spiegel, 1965).

In section 2 we solved this problem in terms of ‘almost’ analytical functions. As an interesting additional result, it appeared that the non-analytical solution is odd with respect to the y-direction of the problem. This offers the opportunity to construct strip-like models of the cochlear scalae by application of the method of images. We applied this method and determined properties of the pressure at the membrane. It appears that the level difference between the levels before and after the point of resonance is about half the same difference in the analytical case. In spite of this reduction, the difference is still very large. Beyond the point of resonance the phase function tends towards a constant value. This effect depends on both the number of images that is applied and the height of the strip. When a limited number of images is applied or when the (scaled) height is sufficiently small the slope of the phase is negative. Then there is a wave that travels to the point of resonance.

It holds that before the point of resonance the amplitude of the pressure and its phase are comparable with the same quantities in the analytical case. Therefore, for numerical applications it is sufficient to restrict ourselves to the original analytical solution of the membrane condition.

References

Békésy, G. von (1928): "Zur Theorie des Hörens: Die Schwingungsform der Basilarmembran", *Physik. Zeits.* **29**: pp. 793-810.

Békésy, G. von (1960): *Experiments in hearing*, McGraw-Hill Co., New York.

Boer, E. de (1979): "Short-wave world revisited: Resonance in a two-dimensional cochlear model", *Hearing Research* **1**: pp. 253-281.

Boer, E. de (1984): "Auditory Physics. Physical principles in hearing theory, II", *Phys. Rep.* **105**.

Dijk, J. S. C. van (2001): *Mechanical aspects of hearing,* Academic Dissertation, University of Amsterdam.

Lighthill, M. J. (1978): *Waves in fluids*, Cambridge University Press, Cambridge.

Lighthill, M. J. (1981): "Energy flow in the cochlea", *J. Fluid. Mech.* **106**: pp. 149–213.

Ranke, O. F. (1931): *Die Gleichrichter-Resonanztheorie,* Lehman, Munchen.

Ranke, O. F. (1942): "Das Massenverhältnis zwischen Membran und Flüssigkeit im Innenohr", *Akustische Zeitschrift* **7**: pp. 1–11.

Rhode, W. S. (1971): "Observations of the vibration of the basilar membrane in the squirrel monkey using the Mössbauer technique", *J. Acoust. Soc. Am.* **49**: pp. 1218–1231.

Rhode, W. S. and Robles, L. (1973): "Evidence from Mössbauer experiments for nonlinear vibration in the cochlea", *J. Acoust. Soc. Am.* **54**: pp. 268(A).

Ruggero, M. A., Rich, C. R., Recio, A., Narayan, S. and Robles, L. (1997): "Basilar membrane responses to tones at the base of the chinchilla cochlea", *J. Acoust. Soc. Am.* **101**: pp. 2151–2163.

Schroeder, M. R. (1975): "Models of hearing", *Proc. IEEE,* **63**: pp. 1332–1350.

Siebert, W. M. (1974): "Ranke revisited - a simple short wave cochlear model", *J. Acoust. Soc. Am.* **56**: pp. 594–600.

Spiegel, M. R. (1964): *Laplace transform,* Schaum's Outline Series, McGraw-Hill, New York.